Lecture 25

Last comments on 15.3:

Let D be a region in $\mathbb{R}^2$:

- $\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

- $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$ (c a constant)

- If $f(x,y) \geq g(x,y)$ for all (x,y) in D:

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

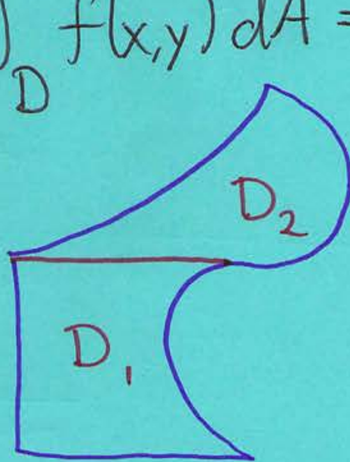
- If $m \leq f(x,y) \leq M$ for all (x,y) in D:

$$mA(D) \leq \iint_D f(x,y) \leq MA(D)$$

- If $D = D_1 \cup D_2$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

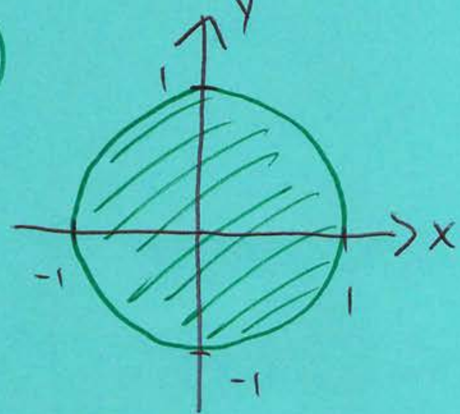
e.g.:

$D =$  $= D_1 \cup D_2$

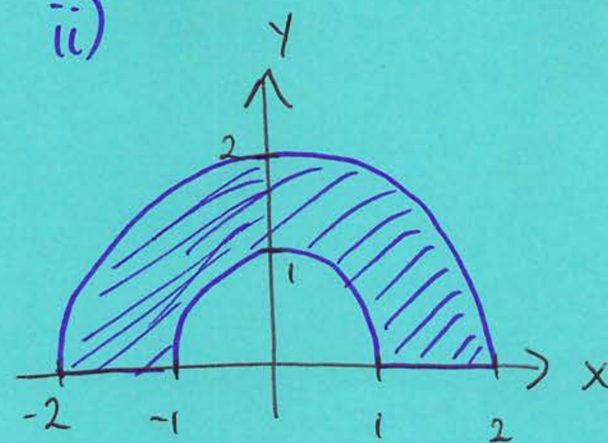
15.4 - Double Integrals in Polar Coordinates

Suppose we want to integrate over a region D, where D is one of the following regions:

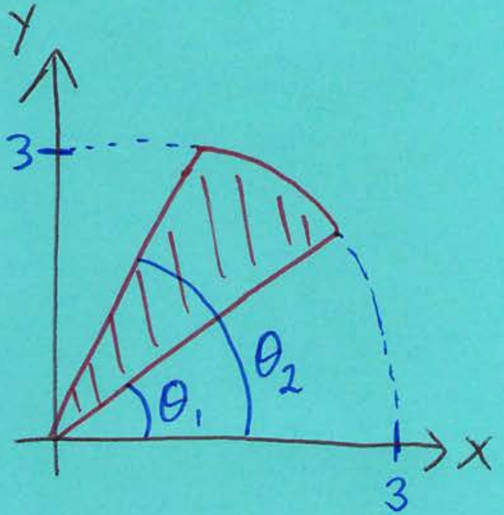
i)



ii)



iii)



The disk might be feasible in cartesian coordinates, but the others certainly aren't. This is because they're difficult to describe with cartesian coordinates.

However, these regions are all very natural in polar coordinates (they're sometimes called <u>polar rectangles</u>).

i) $D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

ii) $D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

iii) $D = \{(r,\theta) \mid 0 \le r \le 3,\ \theta_1 \le \theta \le \theta_2\}$

Recall some essential equations from polar coordinates

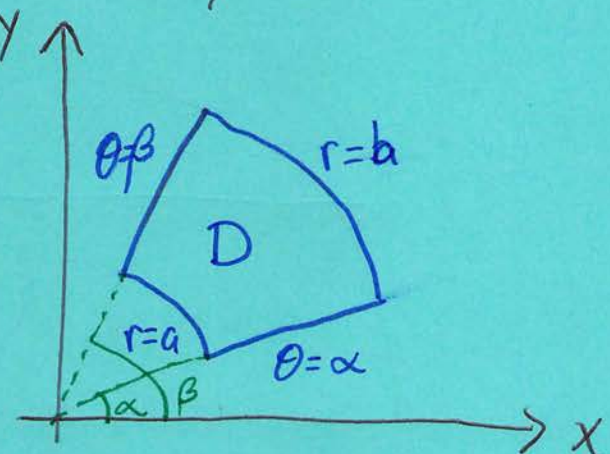
$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \left(\frac{x}{y} = \tan\theta\right)$$

This presents a case for integrating in polar coordinates. So, how do we compute

$$\iint_D f(x,y)\, dA$$

in polar coordinates?

Let's say D is the sector:

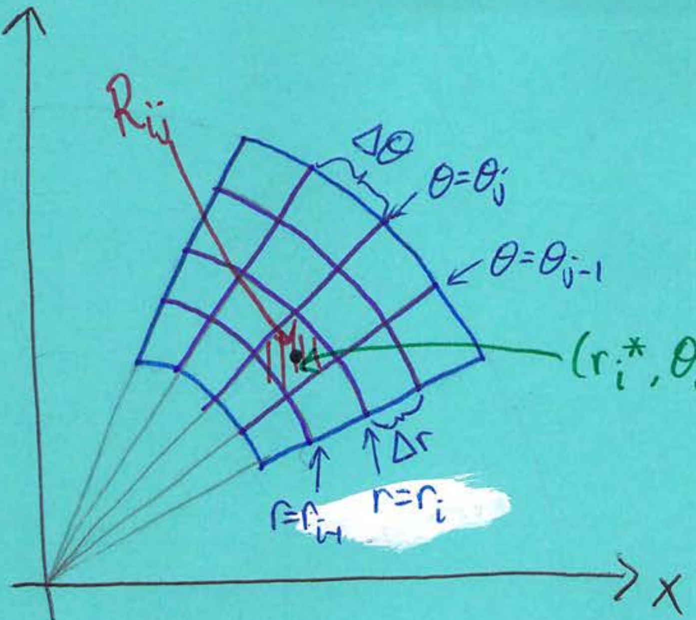


This is a rectangle in polar coordinates since

$$D = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

Since we want an integral with $dr\,d\theta$ or $d\theta\,dr$, let's find a Riemann sum with Δr and $\Delta\theta$ in it.

First, we divide D into tiny polar rectangles:



$$r_i^* = \frac{r_{i-1}+r_i}{2}$$

$$\theta_j^* = \frac{\theta_{j-1}+\theta_j}{2}$$

$$\Delta r = r_i - r_{i-1}$$

$$\Delta\theta = \theta_j - \theta_{j-1}$$

(r_i^*,θ_j^*) is the "center" of the rectangle R_{ij}

Notice the rectangles get bigger as r_i increases, so the area of R_{ij} gets bigger with i. Now:

$$\Delta A_i = \text{area}(R_{ij}) = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta$$

$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta = \underbrace{\frac{1}{2}(r_i+r_{i-1})}_{r_i^*}\underbrace{(r_i-r_{i-1})}_{\Delta r}\Delta\theta$$

$$= r_i^*\Delta r\Delta\theta$$

We then have:

$$\iint_D f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_i$$

$$= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,r_i^*\Delta r\Delta\theta$$

$$= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Changing a double integral to polar coordinates

If f is continuous on a polar rectangle, D, given by $D=\{(r,\theta)\mid a\le r\le b,\ \alpha\le\theta\le\beta\}$, where $0\le\beta-\alpha\le 2\pi$ (this is to make sure we don't wrap around the circle more than once and double count things), then

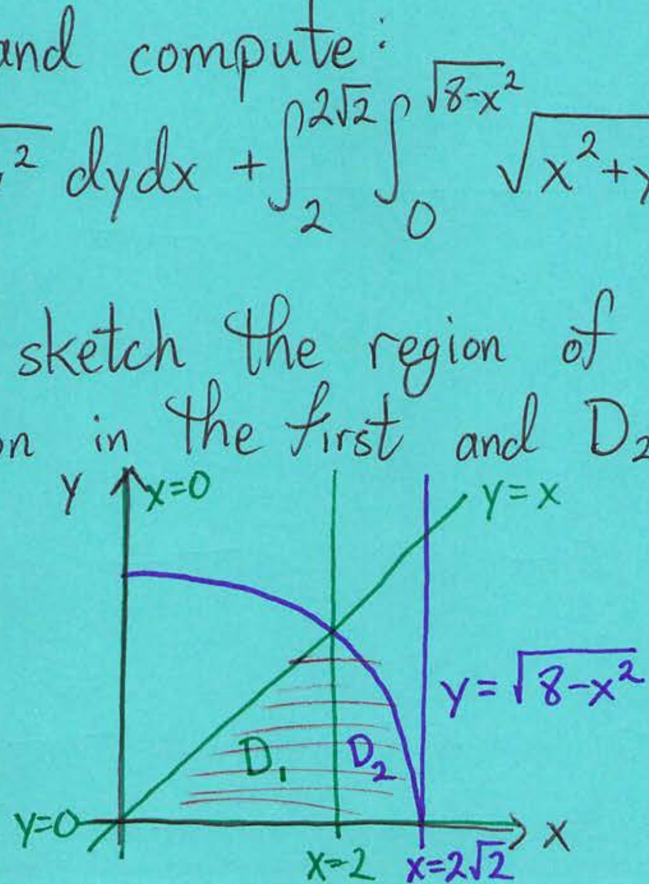
$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_a^b\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

Ex: Combine and compute:

$$\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx+\int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}}\sqrt{x^2+y^2}\,dy\,dx$$

Sol: First, we sketch the region of integration. Let D_1 be the region in the first and D_2 the region in the second:



$D=D_1\cup D_2$

This region is quite nice in polar coordinates:

$D = \{(r,\theta) \mid 0 \le r \le 2\sqrt{2},\ 0 \le \theta \le \frac{\pi}{4}\}$

Also, we need f in terms of r and θ:

Since $f(x,y) = \sqrt{x^2+y^2}$, we have

$$f(r\cos\theta, r\sin\theta) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

So, the integral is

$$\int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{2}} r\,(r\,dr\,d\theta) = \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \int_0^{\pi/4} \frac{r^3}{3}\Big|_0^{2\sqrt{2}} d\theta$$

$$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3}\,d\theta = \frac{4\sqrt{2}\,\pi}{3} \quad \diamond$$

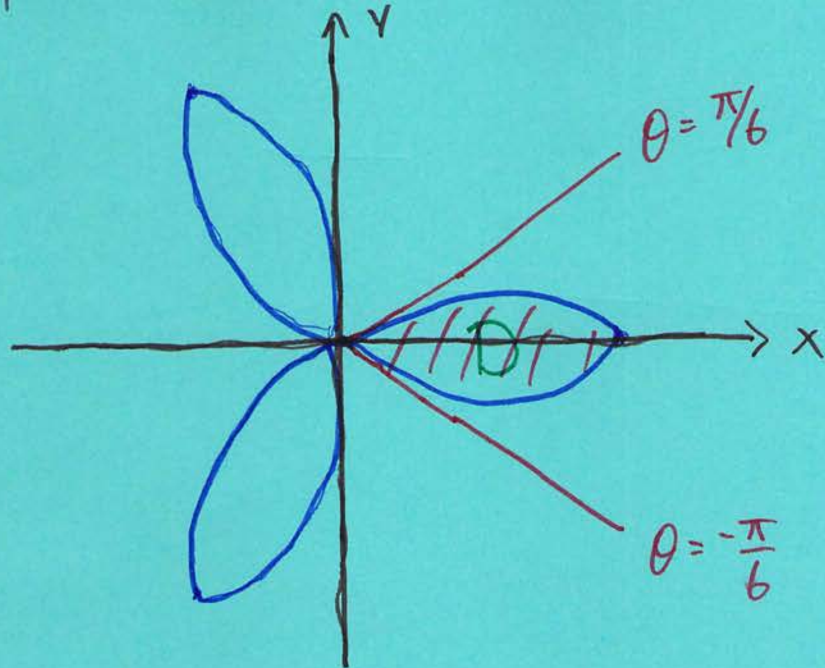
Of course, we need not always integrate over a polar rectangle. If D is the region bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, and $r = k(\theta)$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h(\theta)}^{k(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Let's see an application of this to a familiar problem.

Ex: Find the area of the region enclosed by one petal of the rose $r = \cos 3\theta$.

Sol:



Letting θ vary from $\frac{-\pi}{6}$ to $\frac{\pi}{6}$ gives us one petal.

So, $$A(D) = \iint_D dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2} r^2 \Big|_0^{\cos 3\theta} d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{\cos 6\theta + 1}{4}\, d\theta = \frac{1}{4}\left(\frac{1}{6}\sin 6\theta + \theta\right)\Big|_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4}\left[\left(\frac{1}{6}\sin \pi + \frac{\pi}{6}\right) - \left(\frac{1}{6}\sin(-\pi) + \frac{-\pi}{6}\right)\right] = \frac{\pi}{12}$$

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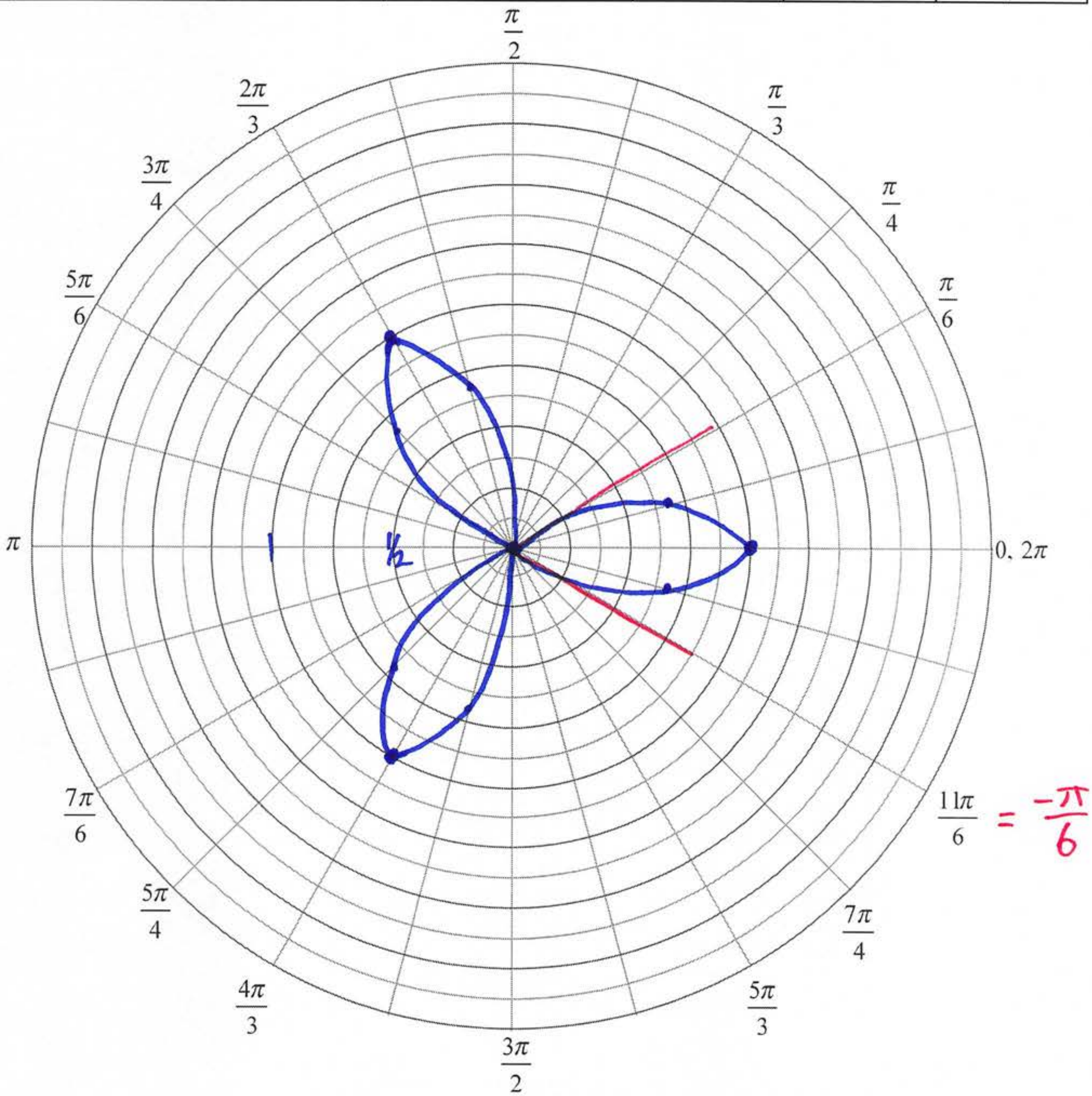
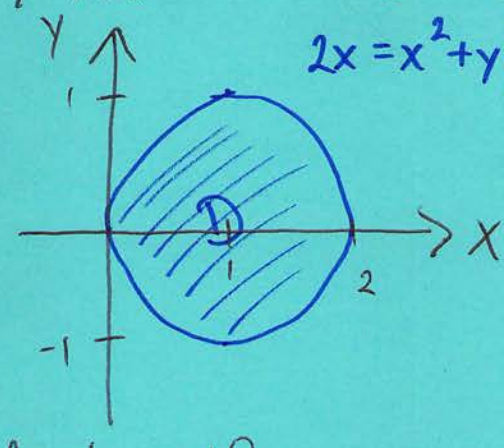


$r = \cos 3\theta,\ 0 \le \theta \le \pi$, $r = 0$ at $\theta = \frac{\pi}{6} + \frac{n\pi}{3}$

Ex: Find the volume of the solid bounded by $z=x^2+y^2$, $2x=x^2+y^2$, and the xy-plane.

Sol: $z=x^2+y^2$ is a paraboloid with minimum at the origin. $2x=x^2+y^2 \Leftrightarrow (x-1)^2+y^2=1$, so a cylinder with axis parallel to the z-axis and passing through (1,0). The graphs of these can be found in the mathematica code. We see that the base of this solid is the disk with boundary $2x=x^2+y^2$ in the xy-plane. We wish to compute

$$Vol=\iint_D (x^2+y^2)\,dA$$ where D is



In polar coordinates, this region is:

$$2r\cos\theta = 2x = x^2+y^2 = r^2 \Leftrightarrow r = 2\cos\theta$$

Now, any interval of length π will graph this circle, so let $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, a symmetric interval.

$$\text{Vol} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\left(16\cos^4\theta\right) d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta \overset{\left(\substack{\cos^4\theta \text{ is} \\ \text{even}}\right)}{=\!=\!=} 8\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 8\int_0^{\pi/2} \left(\cos^2\theta\right)^2 d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= 2\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]\Big|_0^{\pi/2}$$

$$= 2\left[\left(\frac{3\pi}{4} + \cancel{\sin\pi} + \cancel{\frac{1}{8}\sin 2\pi}\right) - \left(0 + \cancel{\sin 0} + \cancel{\frac{1}{8}\sin 0}\right)\right] = \frac{3\pi}{2}$$

$\Diamond$